UNIT-2 (2nd Part)

14

Thick Cylinders

14.1. INTRODUCTION

While dealing with thin cylinders it is assumed that the circumferential stress is constant over the cross-section of the cylinder and the radial stress is negligible. However, it is found that as the thickness of the cylinder increases in proportion to its diameter, these assumptions are no longer valid. This necessitates a more exhaustive analysis in case of thick cylinders.

It is seen that the error involved in the value of the maximum circumferential stress by using thin cylinder formula is less than 5% in case of cylinders with ratio of their diameter (d) to thickness (t) 20. This gives us a broad classification of thick cylinders as the ones having $\frac{d}{t} < 20$. or $t/d < 1/20$

14.2. LAME'S EQUATIONS

The following assumptions are made :

1. The material of the cylinder is homogeneous and isotropic.

2. Modulii of elasticity for the cylinder material in tension and compression are equal.

3. The plane sections of the cylinder perpendicular to the longitudinal axis remain plane after the cylinder has been subjected to pressure. This assumption is valid at a considerable distance from the ends so that the restraining effect of the ends is negligible there. The length of the cylinder is assumed to be sufficiently large for this to be valid.

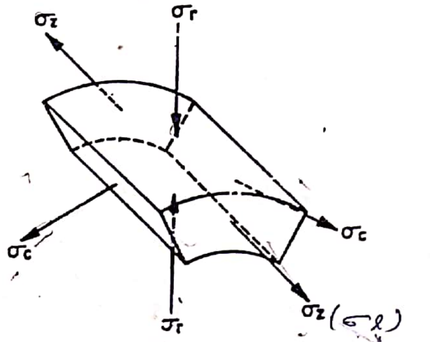

Fig. 14.1

This assumption implies that the longitudinal strain, ε_z, is constant over the cross-section of the cylinder.

In this chapter, circumferential and longitudinal stresses will be assumed positive when tensile, whereas radial stress will be taken to be positive when compressive. From generalized Hooke's law and Fig. 14.1, longitudinal strain, ε_z, is given as,

$$\varepsilon_z = \frac{\sigma_z}{E} + \frac{\nu\sigma_r}{E} - \frac{\nu\sigma_c}{E}$$

[For cylinder not fluid]

$$= \frac{1}{E}[\sigma_z - \nu(\sigma_c - \sigma_r)]$$

Since E, ν are material constants, σ_z is either zero or constant (See Art. 14.3) and ε_z is assumed constant, $(\sigma_c - \sigma_r)$ must also be a constant, say equal to $2a$, *i.e.*,

$$\sigma_c - \sigma_r = 2a$$

or $\theta = l/r$

$$\sigma_c = 2a + \sigma_r \qquad \ldots(14.1)$$

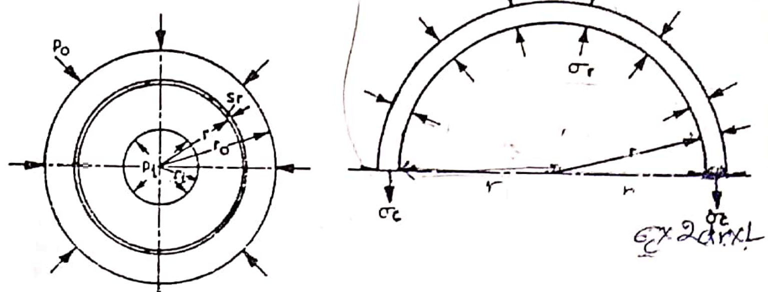

Fig. 14.2

Consider a cylinder of length L, inner radius r_1 and the outer radius r_o. Let this cylinder be subjected to pressure p_i on the inner surface and p_o on the outer surface. Of this cylinder, consider an elementary annular ring of inner radius r and width δr. One half of this ring with various stresses acting on the same (free body diagram) has been shown separately (Fig. 14.2). For equilibrium of this element the algebraic sum of the forces in any direction should be zero. Considering forces in the downward direction,

$$(\sigma_r + \delta\sigma_r)\,[2\,(r + \delta r)\,L] - \sigma_r \cdot 2rL + 2\sigma_c\,\delta r L = 0$$

or

$$\cancel{\sigma_r r} + \sigma_r \cdot \cancel{\delta r} + r\,\delta\sigma_r + \cancel{\delta\sigma_r \cdot \delta r} - \cancel{\sigma_r r} + \sigma_c \cdot \cancel{\delta r} = 0$$

Neglecting the term $\delta\sigma_r \cdot \delta r$ being very small relative to other terms (being product of two small quantities), and dividing throughout by δr,

$$\sigma_r + r\frac{\delta\sigma_r}{\delta r} + \sigma_c = 0$$

or, in the limit, $$\sigma_c = -\sigma_r - r\frac{d\sigma_r}{dr} \qquad \text{...(14.2)}$$

Substituting this value of σ_c into Eq. (14.1),

$$-\sigma_r - r\frac{d\sigma_r}{dr} = 2a + \sigma_r$$

or $$-r\frac{d\sigma_r}{dr} = 2(a+\sigma_r)$$

or, $$\frac{d\sigma_r}{a+\sigma_r} = -2\frac{dr}{r}$$

Integrating, $\ln(a+\sigma_r) = -2\ln r + \ln b$

('ln b' being the constant of integration)

$$= \ln\frac{b}{r^2}$$

or, $$a + \sigma_r = \frac{b}{r^2}$$

i.e., $$\boxed{\sigma_r = \frac{b}{r^2} - a} \qquad \text{...(14.3)}$$

Substituting this value of σ_r in Eq. (14.1),

$$\boxed{\sigma_c = \frac{b}{r^2} + a} \qquad \text{...(14.4)}$$

Equations (14.3) and (14.4) are called *Lame's equations* and are the basic equations employed for the determination of stresses in thick cylinders. The constants 'a' and 'b' in these equations can be evaluated with the help of given boundary conditions as described below for different cases.

14.2.1. General case

$$\sigma_r = p_i \quad \text{when} \quad r = r_i$$

and $$\sigma_r = p_o \quad \text{when} \quad r = r_o$$

Substituting these in Eq. (14.3),

$$p_i = \frac{b}{r_i^2} - a \text{ and } p_o = \frac{b}{r_o^2} - a$$

$$\therefore \quad p_i - p_o = \frac{b}{r_i^2} - \frac{b}{r_o^2} = b\left(\frac{r_o^2 - r_i^2}{r_i^2 r_o^2}\right),$$

$$\therefore \quad \boxed{b = (p_i - p_o)\frac{r_i^2 r_o^2}{r_o^2 - r_i^2}}$$

and $$a = \frac{b}{r_i^2} - p_i = (p_i - p_o)\frac{r_o^2}{r_o^2 - r_i^2} - p_i$$

$$= \frac{p_i r_o^2 - p_o r_o^2 - p_i r_o^2 + p_i r_i^2}{r_o^2 - r_i^2}.$$

$$\boxed{a = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}}$$

Substituting values of 'a' and 'b' in Eq. (14.3) and (14.4),

$$\sigma_r = \frac{b}{r^2} - a$$

$$= (p_i - p_o)\frac{r_i^2 r_o^2}{r^2(r_o^2 - r_i^2)} - \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

$$= \frac{r_i^2 r_o^2}{r^2}\,\frac{p_i - p_o}{r_o^2 - r_i^2} - \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

and $$\sigma_c = \frac{b}{r^2} + a$$

$$= \frac{r_i^2 r_o^2}{r^2}\,\frac{p_i - p_o}{r_o^2 - r_i^2} + \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

i.e., $$\sigma_{rc} = \frac{r_i^2 r_o^2}{r^2}\,\frac{p_i - p_o}{r_o^2 - r_i^2} \mp \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \qquad \text{...(14.5}$$

14.2.2. Thick cylinder subjected to internal pressure only

$$\sigma_r = p_i \quad \text{when} \quad r = r_i$$

and $$\sigma_r = 0 \quad \text{when} \quad r = r_o$$

Substituting these in Eq. (14.3),

$$p_i = \frac{b}{r_i^2} - a \quad \text{and} \quad 0 = \frac{b}{r_o^2} - a$$

$$\therefore \quad p_i - 0 = \frac{b}{r_i^2} - \frac{b}{r_o^2} = b\left(\frac{r_o^2 - r_i^2}{r_i^2 r_o^2}\right)$$

$$\therefore \quad \boxed{b = p_i\frac{r_i^2 r_o^2}{r_o^2 - r_i^2}}$$

and $$\boxed{a = \frac{b}{r_o^2} = p_i\frac{r_i^2}{r_o^2 - r_i^2}}$$

Substituting values of 'a' and 'b' in Eqs. (14.3) and (14.4),

$$\sigma_r = \frac{b}{r^2} - a$$

$$= \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r^2} - 1\right]$$

and

$$\sigma_c = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r^2} + 1\right]$$

i.e.,

$$\sigma_{r_c} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r^2} \mp 1\right] \qquad \text{...(14.6)}$$

It is seen that σ_c is maximum when r minimum, *i.e.* when $r = r_i$

$\therefore$

$$\sigma_{c_{max}} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_i^2} + 1\right]$$

$$= p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \qquad \text{...(14.7)}$$

$$= p_i \frac{k^2 + 1}{k^2 - 1}, \quad \text{where } k = \frac{r_o}{r_i} \qquad \text{...(14.8)}$$

Also, at $r = r_o$,

$$\sigma_c = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_o^2} + 1\right]$$

$$= p_i \frac{2r_i^2}{r_o^2 - r_i^2} = p_i \frac{2}{k^2 - 1} \qquad \text{...(14.9)}$$

Also, σ_r is maximum at $r = r_i$

$\therefore$

$$\sigma_{r_{max}} = p_i$$

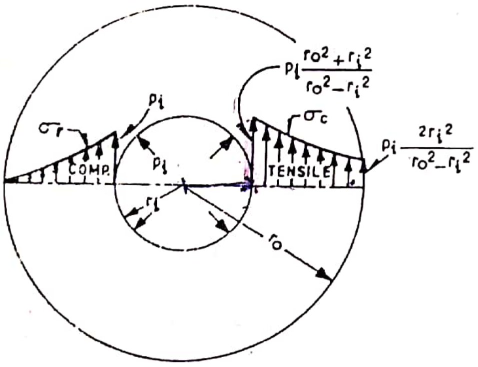

Fig. 14.3

$\therefore$ At the *inner* surface,

$$\frac{\sigma_c}{\sigma_r} = \frac{r_o^2 + r_i^2}{r_0^2 - r_i^2} = \frac{k^2 + 1}{k^2 - 1} \qquad \text{...(14.10)}$$

Thus the distribution of radial and circumferential stresses is as shown in Fig. 14.3. *It may be noted that maximum stress in the cylinder is the hoop stress at the inner radius.*

14.2.3. Thick cylinder subjected to external pressure only

$$\sigma_r = 0 \quad \text{when } r = r_i$$

and

$$\sigma_r = p_o \quad \text{when } r = r_o$$

Substituting these in Eq. (14.3) and (14.4),

$$0 = \frac{b}{r_i^2} - a \quad \text{and} \quad p_o = \frac{b}{r_o^2} - a$$

$\therefore$

$$-p_o = \frac{b}{r_i^2} - \frac{b}{r_o^2} = b\left(\frac{r_o^2 - r_i^2}{r_i^2 r_o^2}\right)$$

$\therefore$

$$b = -p_o \frac{r_i^2 r_o^2}{r_o^2 - r_i^2}$$

and

$$a = \frac{b}{r_i^2} = -p_o \frac{r_o^2}{r_o^2 - r_i^2}$$

Substituting values of 'a' and 'b' in Eqs. (14.3) and (14.4),

$$\sigma_r = \frac{b}{r^2} - a$$

$$= -\frac{p_o r_o^2}{r_o^2 - r_i^2}\left[\frac{r_i^2}{r^2} - 1\right]$$

and

$$\sigma_c = -\frac{P_o r_o^2}{r_o^2 - r_i^2}\left[\frac{r_i^2}{r^2} + 1\right]$$

i.e.,

$$\sigma_{r_c} = -\frac{P_o r_o^2}{r_o^2 - r_i^2}\left[\frac{r_i^2}{r^2} \mp 1\right] \qquad \text{...(14.11)}$$

In Eq. (14.11), since r_i will always be less than r_0, σ_r will be $+$ve, *i.e.*, compressive and σ_c will be $-$ ve, *i.e.* also compressive. It is seen that Eq. (14.11) are similar to Eq. (14.6) for thick cylinder subjected to internal pressure only.

σ_c is maximum when r is minimum, *i.e.* at $r = r_i$

$\therefore$

$$\sigma_{c_{max}} = -\frac{P_o r_o^2}{r_o^2 - r_i^2}\left[\frac{r_i^2}{r_i^2} + 1\right]$$

$$= -p_o \frac{2r_o^2}{r_o^2 - r_i^2} \qquad ...(14.12)$$

$$= -p_o \frac{2k^2}{k^2 - 1}, \quad \text{where } k = \frac{r_o}{r_i} \qquad ...(14.13)$$

Also, at $r = r_o$, $\sigma_c = -\frac{p_o r_o^2}{r_o^2 - r_i^2}\left[\frac{r_i^2}{r_o^2} + 1\right]$

$$= -p_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = -p_o \frac{k^2 + 1}{k^2 - 1} \qquad ...(14.14)$$

Further, σ_r is maximum at $r = r_o$

∴ $\sigma_{r_{max}} = p_o$

∴ At the *outer* surface, $\frac{\sigma_c}{\sigma_r} = \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = \frac{k^2 + 1}{k^2 - 1}$ ...(14.15)

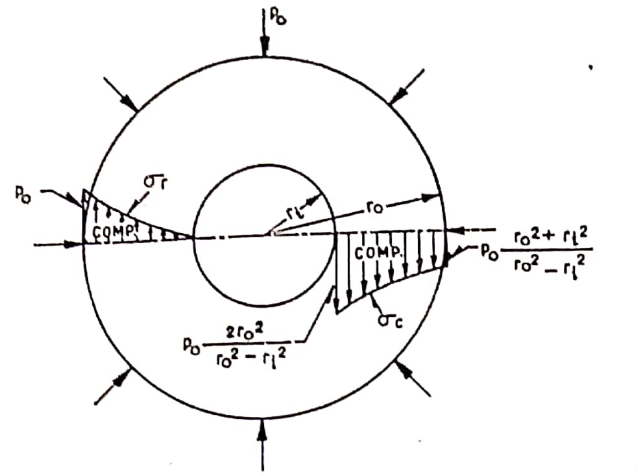

Fig. 14.4

Thus the distribution of radial and circumferential stresses in this case is shown in Fig. 14.4. ***It is seen that maximum stress in the cylinder in this case also, is the hoop stress at the inner radius.***

14.2.4. Thick cylinder with no inner hole

When such a cylinder ($r_i = 0$), the common example of which is shaft, is subjected to external pressure, constant 'b' in Eq. (14.3) and (14.4) has to be

zero, to avoid stresses being infinite for any magnitude of external pressure. Thus from Eq. (14.3) and (14.4),

$$\sigma_r = -a \text{ and } \sigma_c = +a$$

∴ $\boxed{\sigma_c = -\sigma_r = a}$ ...(14.16)

which means both the radial as well circumferential stresses are uniformly distributed and each is equal in magnitude to the external radial pressure. Further *both will be compressive in nature.*

14.3. LONGITUDINAL STRESS

(*a*) If the cylinder is *not closed* at the ends (*e.g.* gun barrel) or when the pressure is retained by a piston at one or both ends of the cylinder (Fig. 14.5), the longitudinal (*i.e.* axial) stress is zero,

i.e. $\sigma_z = 0$ ...(14.17 *a*)

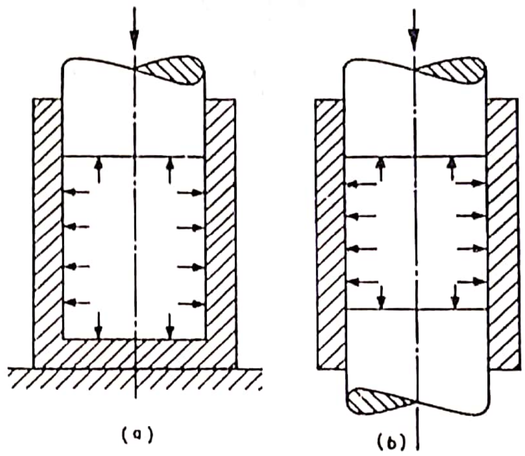

Fig. 14.5

(*b*) If, however, the cylinder is *closed at the ends* by caps (Fig. 14.6), it will be subjected to stress in the longitudinal direction also. Thus in the general case of a cylinder subjected to pressures, both internal and external, the longitudinal stress is given by,

$$\sigma_z = \frac{p_i \pi r_i^2 - p_o \pi r_o^2}{\pi (r_o^2 - r_i^2)}$$

$$\boxed{\sigma_z = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}} \qquad ...(14.17\ b)$$

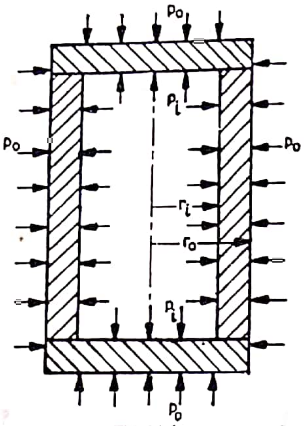

Fig. 14.6

In case of internal pressure only, therefore,

$$\sigma_z = p_i \frac{r_i^2}{r_o^2 - r_i^2} \quad \text{...(14.17 c)}$$

$$= p_i \frac{1}{k^2 - 1} \quad \text{...(14.17 } d\text{)}$$

(*c*) In case, the cylinder is built-in between rigid end supports, axial strain, *i.e.* $\varepsilon_z = 0$. In that case,

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_c - \sigma_r) = 0$$

or

$$\boxed{\sigma_z = \nu\,(\sigma_c - \sigma_r)} \quad \text{...(14.17 } e\text{)}$$

(where σ_c is tensile and σ_r is compressive)

14.4. PRINCIPAL AND SHEAR STRESSES

Since there is no torque acting on the cylinder, the stresses σ_r, σ_c and σ_z are the three principal stresses. Out of these σ_r is compressive and σ_c is always greater than σ_z. Therefore, we can write that,

$$\sigma_1 = \sigma_c, \quad \sigma_2 = \sigma_z \text{ and } \sigma_3 = \sigma_r$$

Therefore maximum shear stress, τ_{max}, is given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

$$= \frac{\sigma_c - (-\sigma_r)}{2} \quad (\because \ \sigma_r \text{ is compressive})$$

$$= \frac{\sigma_c + \sigma_r}{2}$$

$$= \frac{1}{2}\left[\left(\frac{b}{r^2} + a\right) + \left(\frac{b}{r^2} - a\right)\right]$$

[From Eq. (14.3) and (14.4)]

i.e.

$$\boxed{\tau_{max} = \frac{b}{r^2}} \quad \text{...(14.18)}$$

This will be maximum when '*r*' is minimum, *i.e.*, when $r = r_i$. Therefore, *absolute* maximum shear stress occurs on the inner radius and is equal to $\frac{b}{r_i^2}$.

This acts in a direction 45° to σ_c and σ_r

14.5. GRAPHICAL METHOD

It has been explained in Art. 14.2 how the Lame's equations can be solved mathematically for different cases. These equations can also be solved easily with the help of graphical method as explained below for two important cases.

Case. 1. Thick cylinder subjected to internal pressure only

In this case, $\sigma_r = p_i$ when $r = r_i$

and $\sigma_r = 0$ when $r = r_o$

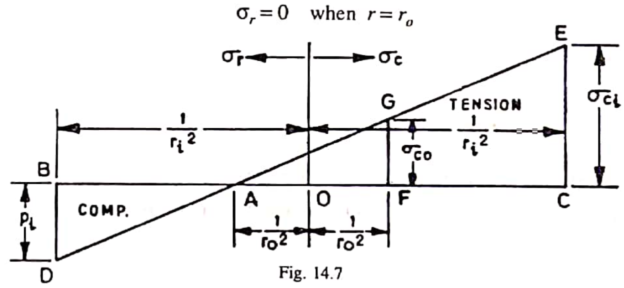

Fig. 14.7

Procedure. (*i*) Draw line *BOC* and mark on it points *B* and *C* at distances $\frac{1}{r_i^2}$ from *O*. Also mark points *A* and *F* at distances $\frac{1}{r_o^2}$ on both sides of *O*.

(*ii*) Draw perpendicular *BD* to line *BC* so that $BD = p_i$.

(*iii*) Join *D* with *A* and produce line *DA* to meet the perpendicular line from *C* to *BC*, at point *E*. Also draw perpendicular to *BC* from point *F*, which meets line *DE* at *G*.

(*iv*) Then $CE = \sigma_{ci}$ *i.e.*, the circumferential stress at the inner surface and $FG = \sigma_{co}$, *i.e.* the circumferential stress at the outer surface. These can be easily measured. Stresses above the datum line *BC* are tensile while those below it are compressive.

(v) In the same way it is seen that if any two quantities out of $\sigma_{ri}, \sigma_{ro}, \sigma_{ci}, \sigma_{co}$ are known, the other two stress values can be determined by proceeding in a similar way.

Proof. Consider similar triangles ABD and AEC.

Then,
$$\frac{BD}{CE}=\frac{BA}{AC}$$

or
$$\frac{p_i}{\sigma_{ci}}=\frac{\frac{1}{r_i^2}-\frac{1}{r_o^2}}{\frac{1}{r_o^2}+\frac{1}{r_i^2}}=\frac{r_o^2-r_i^2}{r_o^2+r_i^2}$$

$\therefore$
$$\sigma_{ci}=p_i\frac{r_o^2+r_i^2}{r_o^2-r_o^2},$$

which is the same as Eq. (14.7).

Considering similar triangles ABD and AFG,
$$\frac{BD}{FG}=\frac{BA}{AF}$$

or
$$\frac{p_i}{\sigma_{co}}=\frac{\frac{1}{r_i^2}-\frac{1}{r_o^2}}{\frac{1}{r_o^2}+\frac{1}{r_o^2}}=\frac{r_o^2-r_i^2}{2r_i^2}$$

$\therefore$
$$\sigma_{co}=p_i\frac{2r_i}{r_o^2-r_i^2},$$

which is the same as Eq. (14.9).

Both σ_{ci} and σ_{co} are tensile as seen from Fig. 14.7.

Case 2. Thick cylinder subjected to external pressure only

In this case, $\sigma_i=0$ when $r=r_i$

and $\sigma_r=p_o$ when $r=r_o$

Procedure. In an exactly similar way as described in the previous case, make AD perpendicular to BC at A and equal to p_o. Join B with D and produce BD so that the perpendiculars to BC at points F and C meet BD produced at points G and E respectively. Then CE and FG are σ_{ci} and σ_{co} respectively.

Proof. Considering similar triangles ABD and CBE,
$$\frac{AD}{CE}=\frac{BA}{BC}$$

or
$$\frac{p_o}{\sigma_{ci}}=\frac{\frac{1}{r_i^2}-\frac{1}{r_o^2}}{\frac{1}{r_i^2}+\frac{1}{r_i^2}}=\frac{r_o^2-r_i^2}{2r_o^2}$$

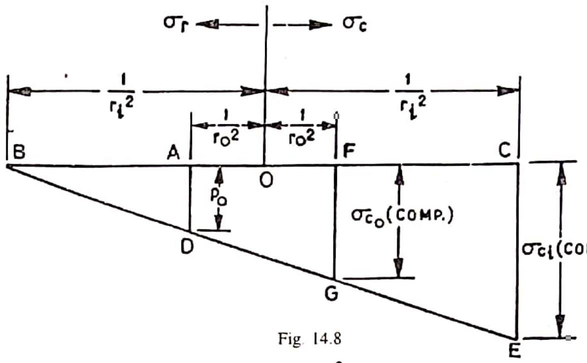

Fig. 14.8

$\therefore$
$$\sigma_{ci}=p_o\frac{2r_o^2}{r_o^2-r_i^2},$$

which is the same as Eq. (14.12).

σ_{ci} is compressive as seen from Fig. 14.8.

Considering similar triangles ABD and FBG,
$$\frac{AD}{FG}=\frac{BA}{BF}$$

or
$$\frac{p_o}{\sigma_{co}}=\frac{\frac{1}{r_i^2}-\frac{1}{r_o^2}}{\frac{1}{r_i^2}+\frac{1}{r_o^2}}=\frac{r_o^2-r_i^2}{r_o^2+r_i^2}$$

$\therefore$
$$\sigma_{co}=p_o\frac{r_o^2+r_i^2}{r_o^2-r_i^2},$$

which is the same as Eq. (14.14).

In is further seen from Fig. 3.8 that σ_{co} is compressive.

14.6. VARIATION OF RADIAL AND HOOP STRESSES WITH THICKNESS

We shall now consider the effect of decreasing or increasing the cylinder thickness on σ_c and σ_r in the case of a cylinder subjected to internal pressure only.

14.6.1. Effect of decreasing the cylinder thickness

(i) Let $r_o=r_i+t$

Then from Eq. (14.7),
$$\sigma_{ci}=p_i\frac{(r_i+t)^2+r_i^2}{(r_i+t)^2-r_i^2}$$
$$=p_i\frac{(r_i+t)^2+r_i^2}{(2r_i+t)\,(t)}$$

Also from Eq. (14.9), $\sigma_{co} = p_i \dfrac{2r_i^2}{r_o^2 - r_i^2}$

$$= p_o \frac{2r_i^2}{(r_i + t)^2 - r_i^2} = p_i \frac{2r_i^2}{(2r_i + t)(t)}$$

$\therefore$ When 't' is small, $\sigma_{ci} \cong p_i \dfrac{2r_i^2}{2r_i t} = \dfrac{p_i r_i}{t}$

and $\sigma_{co} \cong p_i \dfrac{2r_i^2}{2r_i t} = \dfrac{p_i r_i}{t}$

$\therefore$ For small value of t, $\sigma_{ci} = \sigma_{co} = \dfrac{p_i r_i}{t}$,

which is the same value as obtained for the thin cylinder case directly.

(*ii*) Let $d = 2r_i$

Then from (*a*), $\sigma_{c_{max}} = \sigma_{ci} = p_i \dfrac{\left(\dfrac{d}{2} + t\right)^2 + \left(\dfrac{d}{2}\right)^2}{(d + t)(t)}$

$$= p_i \frac{(d + 2t)^2 + d^2}{4t(d + t)}$$

$$= p_i \frac{\left[\left(\dfrac{d}{t} + 2\right)^2 + \left(\dfrac{d}{t}\right)^2\right]}{\left[4\left(\dfrac{d}{t} + 1\right)\right]}$$

If $\dfrac{d}{t} = m$,

$$\frac{\sigma_{c_{max}}}{p_i} = \frac{(m + 2)^2 + m^2}{4(m + 1)}$$

Also from thin cylinder theory, where σ_c is assumed uniform throughout the section,

$$\frac{\sigma_c}{p_i} = \frac{d}{2t} = \frac{1}{2}\left(\frac{d}{t}\right) = \frac{m}{2}$$

$\therefore$ Error $= \dfrac{(m + 2)^2 + m^2}{4(m + 1)} - \dfrac{m}{2} = \dfrac{m + 2}{2(m + 1)}$

$\therefore$ Percentage error $= \dfrac{\dfrac{m + 2}{2(m + 1)}}{\dfrac{(m + 2)^2 + m^2}{4(m + 1)}} \times 100$

$$= \frac{2(m + 2)}{(m + 2)^2 + m^2} \times 100$$

For $m = 20$,

Percentage error $= \dfrac{2(20 + 2)}{(20 + 2)^2 + (20)^2} \times 100 = 4.977$

Thus it is seen that the error in the value of $\dfrac{\sigma_c}{p_i}$ by assuming it as a thin cylinder is less than 5%, when $\dfrac{d}{t}$ ratio is 20. This is the reason why cylinders with $\dfrac{d}{t}$ ratio more than 20 are considered thin cylinders.

14.6.2. Effect of increasing the cylinder thickness

From Eq. (14.6), when $t \to \infty$, *i.e.* when r_i can be neglected in comparison to r_o,

$$\sigma_{r_c} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r^2} \mp 1\right]$$

$$\cong \frac{p_i r_i^2}{r_o^2}\left[\frac{r_o^2}{r^2}\right] \qquad \text{(Since } r_o^2 \mp r^2 \cong r_o^2\text{)}$$

$$= p_i \frac{r_i^2}{r^2},$$

which is shown diagrammatically in Fig. 14.9.

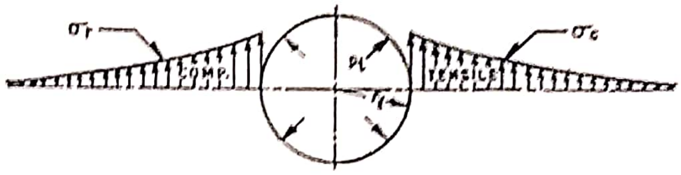

Fig. 14.9

This implies that for a cylinder of very large thickness,

(*a*) At any point, the radial and the circumferential stresses are equal.

(*b*) If $\sigma_z = 0$, state of pure shear exists everywhere in the cylinder.

(*c*) Both σ_r and σ_c are inversely proportional to r^2. Thus at $r = 4r_i$, stresses are only $\dfrac{1}{16}$ of their maximum value, which means that a cylinder of $k = \dfrac{r_o}{r_i} > 4$ may be reasonably assumed to be a *cylinder of infinite thickness*.

The implication of this is that if the shape of the outer boundary of the cylinder cross-section is not uniform *i.e.*, circular but if all points on this

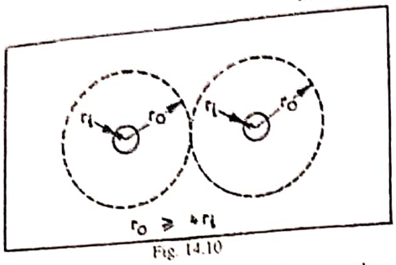

Fig. 14.10

non-uniform boundary are at a distance of at least $4r_i$ from the centre, the shape of such non-uniform boundary will not matter. Thus for example in Fig. 14.10, though the outer shape is rectangular, yet it may be treated cylindrical as shown in dotted lines.

Example 14.1. *The internal diameter of the cylinder of a hydraulic jack is 100 mm. It is required to operate upto a pressure of 10 MPa. If the yield strength of the cylinder material is 40 MPa, calculate the wall thickness.*

Sol. Lame's equations are,

$$\sigma_r = \frac{b}{r^2} - a$$

$$\sigma_c = \frac{b}{r^2} + a$$

Here it is given that, $\sigma_r = 10$ MPa

$= 10$ N/mm^2 at $r = 50$ mm

$\sigma_r = 0$ at $r = r_o$ i.e. outer radius

and $\sigma_c = 40$ MPa

$= 40$ N/mm^2 at $r = 50$ mm

$$\therefore \quad 10 = \frac{b}{50^2} - a$$

$$40 = \frac{b}{50^2} + a$$

Solving these, we get $a = 15$

$b = 62500$

Also $$0 = \frac{b}{r_o^2} - a$$

$$= \frac{62500}{r_o^2} - 15$$

or $r_o = 64.55$ mm

$\therefore$ Thickness $= 64.55 - 50 =$ **14.55 mm**

Example 14.2. *The pressures at the outer and the inner surface of a cylinder are 20 MPa (gauge) and atmospheric (i.e. zero gauge) respectively. If the hoop stress at the inner surface is 60 MPa (compressive), determine its value at the outer surface.*

Sol. Substituting the given boundary conditions in the Lame's equations,

$$20 = \frac{b}{r_o^2} - a \qquad \text{...(i)}$$

$$0 = \frac{b}{r_i^2} - a \qquad \text{...(ii)}$$

and $$-60 = \frac{b}{r_i^2} + a \qquad \text{...(iii)}$$

where r_i and r_o are the inner and the outer radii of the cylinder respectively.

From (*ii*) and (*iii*), we get,

$$a = -30$$

and $$\frac{b}{r_i^2} = a = -30$$

From (*i*), $$\frac{b}{r_o^2} = 20 + a = 20 - 30 = -10$$

$\therefore$ Hoop stress at the outer surface,

$$\sigma_{c_{r=r_o}} = \frac{b}{r_o^2} + a$$

$$= -10 - 30$$

$$= -40 \text{ N/mm}^2 = \mathbf{40\ MPa\ (comp.)}$$

Example 14.3. *In a thick cylinder with internal pressure of 6 MPa, the circumferential stress at the outside surface is 20 MPa. Calculate the circumferential stress at the inside surface and at the point where the radial stress is 3 MPa. Find out the longitudinal stress if the cylinder is closed at the ends and the inside diameter is 200 mm.*

Sol. The given conditions are,

$\sigma_r = 6$ MPa

$= 6$ N/mm^2 at $r = r_i$

$\sigma_r = 0$ at $r = r_o$

and $\sigma_c = 20$ N/mm^2 at $r = r_o$

Substituting these in the Lame's equations,

$$6 = \frac{b}{r_i^2} - a \qquad \text{...(i)}$$

Sol. Let $\quad p_i =$ internal pressure

Then $\quad \sigma_c\big|_{r=r_i} = 1.6\, p_i$

Now, $\quad \sigma_r = p_i$ at $r = r_i$

and $\quad = 0$ at $r = r_i + t = r_o$

$$\therefore \quad p_i = \frac{b}{r_i^2} - a$$

$$0 = \frac{b}{r_o^2} - a$$

which give, $\quad b = p_i \dfrac{r_i^2 r_o^2}{r_o^2 - r_i^2}$ and $a = p_i \dfrac{r_i^2}{r_o^2 - r_i^2}$

$$\therefore \quad \sigma_c\big|_{r=r_i} = \frac{b}{r_i^2} + a = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 1.6\, p_i \text{ (given)}$$

$$\therefore \quad \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 1.6$$

which gives, $\quad \dfrac{r_o}{r_i} = 2.082$

i.e., $\quad \dfrac{r_i + t}{r_i} = 2.082$

or $\quad \dfrac{t}{r_i} = 1.082$

or $\quad \dfrac{t}{d_i} = \mathbf{0.541}$

14.7. STRAINS IN THICK CYLINDER

1. Change of diameter is directly proportional to the change in circumference

$$\text{Circumferential strain} = \frac{\pi(d + \delta d) - \pi d}{\pi d}$$

$$= \frac{\delta d}{d} = \text{diameteral strain.}$$

Therefore, *diameteral strain is always calculated in the direction of* σ_c (circumferential or the hoop stress) (Fig. 14.1) and for our notation, is given by

$$\varepsilon_c = \frac{\sigma_c}{E} + \nu \frac{\sigma_r}{E} - \nu \frac{\sigma_z}{E} = \frac{1}{E}[\sigma_c + \nu(\sigma_r - \sigma_z)] \quad \text{...(14.19)}$$

$\therefore$ For the general case, assuming that the cylinder is closed with caps at its ends, [Substituting for σ_c, σ_r and σ_z from Eq. 14.5 and 14.17 (b),

$$\varepsilon_c = \frac{1}{E}\left[\left\{\frac{r_i^2 r_o^2}{r^2} \cdot \frac{p_i - p_o}{r_o^2 - r_i^2} + \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right\} + \nu \left\{\frac{r_i^2 r_o^2}{r^2} \cdot \frac{p_i - p_o}{r_o^2 - r_i^2} - \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right\}\right]$$

$$= \left\{\frac{(1+\nu)}{E} \frac{r_i^2 r_o^2}{r^2} \frac{p_i - p_o}{r_o^2 - r_i^2} + \frac{(1-2\nu)}{E} \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right] \quad \text{...(14.20)}$$

In the case when the axial stress, σ_z, is not present, this reduces to,

$$\varepsilon_c \approx \frac{1+\nu}{E} \frac{r_i^2 r_o^2}{r^2} \frac{p_i - p_o}{r_o^2 - r_i^2} + \frac{1-\nu}{E} \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \quad \text{...(14.21)}$$

2. The *radial strain* (causing radial shift) at any point in the cylinder is calculated in the direction of the radial stress σ_r (Fig. 14.1) and is given by,

$$\varepsilon_r = \frac{1}{E}[\sigma_r + \nu(\sigma_c + \sigma_z)] \quad \text{...(14.22)}$$

3. Volumetric strain, $\varepsilon_v = \dfrac{\text{Change in volume}}{\text{Original volume of the cylinder}} = \dfrac{dV}{V}$

Now volume, $\quad V = \pi r^2 l$

$$\therefore \quad dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial L} dL$$

$$\approx (2\pi r L)\, dr + (\pi r^2)\, dL$$

$$= \pi r^2 L \left[2 \frac{dr}{r} + \frac{dL}{L}\right]$$

$$\frac{dV}{V} \approx 2 \frac{dr}{r} + \frac{dL}{L}$$

i.e., Volumetric strain $\approx 2 \times$ diameteral (*i.e.* circumferential) strain + longitudinal strain

i.e., $\quad \varepsilon_v = 2 \times \varepsilon_c + \varepsilon_z \quad$ ...(14.23)

Example 14.7. *A cast steel cylinder of 600 mm external and 400 mm internal diameters is subjected to an external pressure of 30 MPa. Calculate the decrease in the external diameter of the cylinder. E = 200 GPa, ν = 0.3.*

Sol. $\quad \sigma_r = 30\ \text{N/mm}^2$ when $r = 300$ mm

$\quad = 0$ when $r = 200$ mm

$$\therefore \quad 30 = \frac{b}{300^2} - a \text{ and } 0 = \frac{b}{200^2} - a$$

which give, $\quad b = -216 \times 10^4$ and $a = -54$

$\therefore$ Hoop stress at external radius,